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A REVIEW OF MINER'S RULE AND SUBSEQUENT GENERALIZATIONS FOR CALCULATING EXPECTED FATIGUE LIFE

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Summary

This paper re-examines the physical assumptions which were made by the originators of the Miner-Palmgren Rule for the calculation of fatigue life and cites publications which show that these assumptions, called the linear cumulative damage hypotheses, are contrary to our present knowledge about actual fatigue behavior. However, work is also discussed which provides evidence that Miner's Rule is better on the average in engineering applications than any other rule for fatigue life which has been advanced. The recent technical papers which resolve this supposed contradiction are referenced and the implications of their results explained in full detail.

These papers show that the linear cumulative damage hypothesis is unnecessary in the derivation of the Miner-Palmgren Rule for the calculation of fatigue life by constructing an alternative stochastic model for which this rule does give the expected life. However, using the original Miner-Palmgren Rule disregards the effect of load order. A more realistic model would be a generalization of Miner's Rule which takes into account the knowledge gained from programmed load studies of load order influence. Moreover, it should recognize that actual loading environment itself frequently may be stochastic in nature. The results of a paper which made these assumptions is discussed here.

Some of the implications of combining these results with a statistical law for the variability of fatigue life about its expectation are examined. This law is constructed from a general class of two-parameter distributions including the log-normal and Weibull as special

cases. The assumption that the shape parameter of the fatigue life distribution is a constant of the material which can be determined from prior data is introduced and the sample theory is provided to utilize the data accumulated in small fatigue life tests for details in order to determine this value.

Lastly, we show that one is able to give a definition of scatter factor as a precise concept with a pre-assigned probability of failure; thus it is possible to establish a warranted life in terms of fleet or detail exposure. This may allow the elimination, in the engineering design process, of some of the present conservatism inherent in the use of arbitrary scatter factors which is necessitated by uncertain knowledge and theory.

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Introduction

The basis for the calculation of fatigue life has required knowledge of the fatigue strength of the component, the loading spectrum and the imposed scatter factors. The early methods of analysis established the fatigue strength from a limited number of fatigue tests at various constant stress levels, then obtained flight measurements from which a vibratory load spectrum was derived. Lastly, scatter factors were applied to the test results and a safe life was calculated by means of some rule, usually Miner-Palmgren, using the load spectrum.

In the past, the scatter factors used were somewhat arbitrary and neither the stochastic nature of fatigue strength nor the statistical requirements for adequate testing were fully understood; moreover, there was considerable argumentation about which fatigue life rule should be used. Nevertheless, the determinations made were in the main successful in reducing catastrophic failures to an acceptable level.

The reasons for this success may lie in the adoption of considerable conservatism either in the flight spectrum, in the scatter factors or in the fatigue life rule. But knowledge, both experimental and theoretical, is necessary to illuminate the exact reasons. If this knowledge could be obtained, a refinement of the design process would be possible through the elimination of the conservatism which is now necessitated by our uncertain knowledge and theory. What is needed is to make use of the knowledge of load order influence acquired in programmed load studies, taking into account the influence of sample sizes in the statistical analysis of failure data and the fact that the actual loading environment is stochastic in nature in the calculation of the safe fatigue life.

In the following pages we present a synopsis of some recent work on the important statistical questions which arise in problems of determining fatigue life.

1. A Formula for Expected Life

By the early sixties there had been such a proliferation of mutually incompatible theories for the calculation of fatigue life that several comprehensive studies were undertaken to compare them. Two of these were sponsored by the U.S. Air Force. One, in 1963, was of a theoretical nature, see [1], and the other, completed in 1962, see [2], was an engineering evaluation of the extant methods for the prediction of fatigue life. The conclusion of the second report contained the statement: "The statistical nature of many facets of the fatigue problem precludes hope of any specific fatigue life prediction of a single article. The best that can be achieved is broad comparisons of the expectations of new structures compared with current and past fleet performance... From this study the use of the Linear Cumulative Damage Hypothesis (Miner's Rule) is recommended as best qualified from the standpoint of simplicity, versatility and of sufficient accuracy (in view of other intangibles in the problem) for use in design."

The words "Miner's Rule" and "Linear Cumulative Hypothesis" referred to a result in a paper published in 1945 by M. A. Miner in [3]. This was in fact an independent derivation of a formula given earlier by A. Palmgren in 1924, see [4].

A disadvantage of a "rule" is that the conditions under which it can be used are not stated, usually because they are not known, whereas a mathematical theorem has an explicit hypothesis which always gives sufficient and sometimes necessary conditions.

The original assumptions, under which Miner proved the rule, became known as the Linear Cumulative Damage Hypotheses: to wit,

I. a) Each specimen can absorb the same amount of fatigue damage and when that amount is attained failure occurs.

b) The amount of damage absorbed by the material in any one cycle is determined only by the load during that cycle.

c) The total damage absorbed by the specimen under a sequence of load cycles is equal to the sum of damages absorbed in each cycle during the sequence.

The famous result called "Miner's Rule" is contained in

Theorem 1: (Miner [3])

If hypotheses I hold and there are only k possible load cycles where ν_i equals the number of cycles to failure under repetition of the i^{th} load cycle, then a loading spectrum which contains n_i applications of the i^{th} cycle for $i=1,\ldots,k$ can be repeated ν times until failure where

$$\nu = \frac{1}{\sum_{i=1}^{k} \frac{n_i}{\nu_i}} . \tag{1.1}$$

The contumely which has been heaped upon Miner's Rule, as expressed in (1.1), has been based on the simplistic and unrealistic nature of the hypotheses I. The obvious question is: How can it be that such a rule is in some average sense good (in fact, best of those compared), by empirical verification as in [2], when the hypotheses are known to be false? The answer is just as obvious: The conclusion must be true under more general conditions.

In 1965 a study was begun to try to replace all of the deterministic assumptions made by Miner in I with stochastic ones which were more realistic. The alternative hypotheses, published in 1968 in the *SIAM Journal* in [5], were:

II. i) Fatigue failure, due to the growth and extension of a dominant failure crack, occurs when a (random) initial length W is reached.

ii) The (random) incremental crack extension Z_i during the i^{th} cycle has a distribution depending only upon that cycle.

iii) For each $i=1,2,\ldots$ the random variable Z_i is non-negative and has a distribution with an increasing failure rate. Moreover, the partial sums $S_n = \sum_1^n Z_i$ are statistically independent of W for all $n=1,2,\ldots$.

We now state the relevant conclusion of that investigation as

Theorem 2: (Birnbaum, Saunders [5])

If hypotheses II hold and we let N_i be the (random) number of cycles to failure under repetition of the i^{th} cycle with finite mathematical expectation $\nu_i = EN_i$ for $i=1,\ldots,k$, then the random number of times a spectrum, containing n_i repetitions of the i^{th} cycle, can be applied until failure has finite expectation ν bounded by

$$\frac{1}{\sum_{i=1}^{k} \frac{n_i}{\nu_i}} - 1 \le \nu \le \frac{1}{\sum_{i=1}^{k} \frac{n_i}{\nu_i + 1}} . \tag{1.2}$$

The point is that the expression for ν given in (1.1) lies between the bounds given in (1.2). Moreover, one can begin to see that if Miner's

Rule did in fact predict only the mean, that statistical variation in measurements would naturally occur about this value but on the average it would be correct. Also, we could expect a contribution to the dispersion of the estimate about the true expected life because of sample fluctuation in estimating the ν_i, $i=1,\ldots,k$.

There are only two points that I wish to discuss about II. The first is that the assumption of crack extension being functionally independent of the preceding loads could only be an approximation to reality in the early stages of crack initiation and growth. It is known to be false at later stages, see [6]. The second point of possible contention, namely increasing failure rate (IFR), was justified in [5] by the "rip in the screen door" model which we now repeat.

Consider a macroscopic crack within a material which, to fix ideas, we picture as follows:

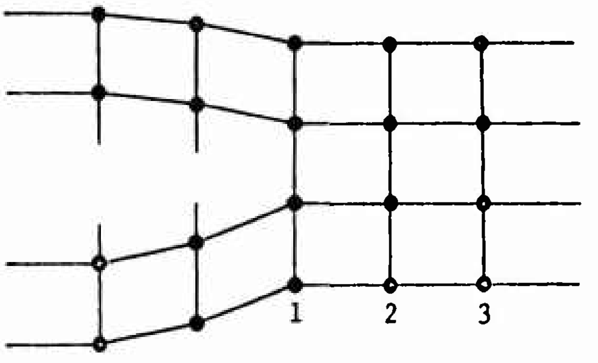

Figure 1

For a given stress imposed let U be the (random) number of bonds broken. Let q_i be the probability that the i^{th} bond is broken given that the $(i-1)^{st}$ bond is broken.

It is intuitively clear that for a giver stress the probability of rupture, given the preceding bond is broken, should decrease the farther away the bond is from the crack tip. But it can be shown, see [5], that the q_n are decreasing if and only if the random variable U is IFR. Thus we conclude that the IFR assumption is intrinsically a natural condition.

We now turn to some results, reported in [8], which require more careful definition and somewhat more mathematical sophistication.

A *load function* is a continuous piecewise linear function on the positive real line, the value of which at any time gives the stress imposed by the deflection of the specimen. Moreover, the slope of such a function changes sign exactly once at the midpoint of each interval of unit length at $(j,j+1)$ for $j=0,1,2,\ldots$. An *oscillation* of a load is the function restricted to an interval of unit length on half of which the function is increasing. (This definition is due to metallurgical opinion that crack growth occurs only during the tension portion of the cycle.) A *load spectrum* is a load function which takes the value zero except on some interval of the form $(0,m)$ where the integer m is called the *duration*. A *programmed load* $\underline{\lambda}$ is the repetition of a spectrum λ, of duration m say, $\underline{\lambda} = (\lambda_1,\lambda_2,\ldots)$ where λ_j is the j^{th} repetition of the spectrum load λ and defined for $t > 0$, $j=0,1,\ldots$ by

$$\lambda_{j+1}(t) = \lambda(-jm+t).$$

By a *load history* (or partial spectrum) of length i, denoted by λ^i, we mean λ restricted in domain to the interval $(0,i)$, with or without subscripts on λ.

A non-negative random variable X is said to be "new better than used in expectation" *(NBUE)* whenever for each $x > 0$ knowing that X exceeds x the expected excess of X over x does not exceed the expectation of X, namely

$$E[X-x|X>x] \leq EX \quad \text{for all} \quad x > 0.$$

This concept was introduced in reliability studies in 1964 in [7].

Using this concept we can formulate a third set of hypotheses:

III. 1° Fatigue failure, due to the initiation and growth of a dominant fatigue crack, occurs when a (random) critical size W is reached.

2° The i^{th} incremental extension during the last oscillation of the history λ_j^i is a non-negative random variable $Z_j^i(\lambda_j^i)$ depending only upon λ_j^i. The $Z(\lambda)$ for all affixes i,j are mutually independent random variables independent of W.

3° The incremental growth random variable $Z(\lambda)$ for all affixes is an NBUE random variable.

4° There exists a finite set of loading oscillations, say $\Omega = \{\omega_0,\ldots,\omega_k\}$, such that for any admissible loading history λ we have an equivalent $\omega_j \in \Omega$, written $\lambda \simeq \omega_j$, for which in distribution $Z(\lambda) = Z(\omega_j)$.

Let us comment on the degree of generality we have postulated. The incremental crack extension may depend upon all the loads previously imposed during that spectrum's repetition as well as the actual propagating load. The incremental growth random variables are assumed to satisfy the very weak NBUE criterion, which is more general than the

IFR class. This means that for a given load, knowing the fatigue damage exceeds a given amount, we conclude that the expected residual fatigue damage from that load is less than the expected damage was from the load before it was applied initially. This would seem to be virtually undeniable. We also assume that there exists a set of loading oscillations Ω to which we refer for fatigue damage assessment whose relations have been measured. These are often presented in what is called the Wöhler diagram or S-N plot.

We can now state a result from [8] concerning programmed loads.

Theorem 3: If the hypotheses III are satisfied, then each programmed repetition of the spectrum λ results in a sequence of independent replications of the random crack growth

$$Y(\lambda) = \sum_{i=1}^{m} Z^i(\lambda^i)$$

which are NBUE. If $N_{Y(\lambda)}$ is the random number of spectra which can be sustained until faiure, it has finite expectation bounded by

$$\frac{1}{\sum_{j=1}^{k} \frac{n_j(\lambda)}{\nu_j}} - 1 \leq EN_{Y(\lambda)} \leq \frac{1}{\sum_{j=1}^{k} \frac{n_j(\lambda)}{\nu_j+1}} \tag{1.3}$$

where

$$n_j(\lambda) = \sum_{i\geq 1} \{\lambda^i \simeq \omega_j\}, \qquad \nu_j = EN_{Y(\omega_j)}, \quad j=1,2,\ldots$$

and $\{\pi\}$ is the indicator function of the relation π being one if true and zero otherwise.

Let Λ denote a random function taking values in the space of admissible load functions $\mathcal{L}$. Assume a *random load* $\underline{\Lambda} = (\Lambda_1, \Lambda_2, \ldots)$ where each Λ_j is an independent replication of the random spectrum Λ.

We now state another result given in [8] which concerns random spectra.

Theorem 4: If hypotheses III are satisfied, then the expected number of random spectra which can be sustained until failure, is bounded below and

$$\frac{1}{\sum_{1}^{k} \frac{En_j(\Lambda)}{\nu_j}} - 1 \leq EN_{Y(\Lambda)} . \qquad (1.4)$$

To obtain an upper bound, analogous to that given in (1.3), appears to be mathematically difficult. In fact we found it necessary to make an additional assumption.

The incremental damage $Z(\lambda)$ during the last oscillation of the spectrum λ has a complementary distribution (unity minus the distribution) we label $R(x:\lambda)$ for $x > 0$. We now make the assumption:

5° For any $x > 0$, $R(x:\lambda)$ is a convex function of λ over the convex space $\mathcal{L}$.

The physical plausibility of this assumption is discussed in [8]. We obtain an upper bound in

Theorem 5: If III and 5° are satisfied, then we have, in the notation of Theorem 4,

$$EN_{Y(\Lambda)} \leq \frac{1}{\sum_{j=1}^{k} \frac{n_j(E\Lambda)}{\nu_j+1}} . \qquad (1.5)$$

For the usual situation where each oscillation of stress would cause failure in between 10^2 and 10^6 repetitions, we conclude that the random number, say N, of times the spectrum Λ formed from such oscillations can be repeated has expectation given approximately by

$$EN = \frac{1}{\sum_{j=1}^{k} \frac{En_j(\Lambda)}{\nu_j}} \tag{1.6}$$

We shall, in what follows, assume that (1.6) holds exactly.

As a final comment, it well may be that the probabilistic structure of Λ which arises in practical applications may contain a form of symmetry so that $En_j(\Lambda)$ is virtually equal to the actual count of oscillations of a particular type and thus would appear to be independent of order. This would account for the closeness of Miner's Rule in the original form to the true value of the expected life. Moreover, the inequality given in (1.4) accounts for conservative tendency in practice. And lastly, we note the formula (1.6) is almost identical to the one given by Freudenthal and Heller in [9] where instead of $En_j(\Lambda)$ they have utilized what they termed "empirical interaction factors".

2. A Connection with Service Life

Fatigue tests for life length are recorded in number of cycles to failure. The distinction between the discrete random variable number of cycles and the continuous random variable life time is not usually maintained since it is often assumed that there is a known relationship of cycles per unit time in service. Let us assume that the fatigue life variability exhibited for a given detail under any loading regime and environment is a non-negative random variable which can be described by the following general class:

A° The observed fatigue life X has an unknown distribution within the two-parameter family, defined for given F, by

$$P[X \leq x] = F[(x/\beta)^{\alpha}] \quad \text{for} \quad x > 0$$

where $\alpha > 0$ is the shape parameter and $\beta > 0$ is the scale parameter (often called the characteristic life).

Note that this formulation includes many of the usual models including the log-normal and the Weibull by proper specification of F.

Of course, we could formulate a model with unknown scale and location parameters by considering $\log X$ as the observable variate and because of the extreme variability in fatigue life observations it is frequently the logarithm which is used in engineering study. However, we prefer to use the former model and there is an easy transformation from one to the other.

Let X_0 be the random variable with distribution F, then

$$X = \beta X_0^{1/\alpha}, \qquad \ell n\, X = \ell n\, \beta + \frac{1}{\alpha}\, \ell n\, X_0. \tag{2.1}$$

We now make assumption

B° The shape parameter α for the distribution of X remains fixed within the family.

Thus we see the variance of the logarithm of service life depends only upon α and the choice of F.

The whole point of this discussion is one should choose statistical procedures which do not depend strongly upon the choice of F. One should make estimates and reach conclusions which are the same for a rather wide class of the choices of F at which nature may arrive (and of which we must ultimately remain ignorant).

The verification of B° represents a non-trivial engineering and statistical task. A document which represents a start in this direction is [10]. It provides specific statistical methods for the treatment of the type of data obtained in fatigue tests for the estimation of α. It also classifies from prior data the conditions of practical concern under which α may be considered constant and then determines this value.

Thus it is possible to determine a model so that sound statistical prediction can be based on one observation which can be used to estimate β_i. We shall be concerned here only with A°, B° in so far as they relate to the assumptions of the previous section. Recall Λ denotes a random spectrum (of random duration M) where all the oscillations were assumed to be of the same duration. For the representation of service life this is, of course, a fiction of mathematical convenience.

Let $U(t)$ be a stochastic process which increases linearly from one integer value to another. The length of time between each integer represents the time between each oscillation. Hence $\Lambda[U(t)]$, $t > 0$ represents an actual random loading as encountered in service with random changes in both amplitude and frequency with t denoting real time. The service time to complete the spectrum Λ is random and is expressed by $L = U^{-1}(M)$, the random time to complete M oscillations.

Let T denote the total service life under usage $\Lambda(U)$, then

$$T = \sum_{j=1}^{N} L_j$$

where L_j, the length of j^{th} repetition of the spectrum, for $j=1,2,...,$ are independent and identically distributed independent of N. That is to say, failure is a result of the number and type of oscillations and not of the length of time between (at least for the range of frequencies we are considering). Hence from Theorem 4°

$$ET = (EN)(EL) = \frac{EL}{\sum_{1}^{k} \frac{En_j(\Lambda)}{\nu_j}} . \tag{2.2}$$

But from (2.1), $ET = \beta g(\alpha)$ where g is a functional of F. But also from A° and B° we see $\nu_j = \beta_j g(\alpha)$ is the expected life and β_j is the characteristic life under repetition of oscillations of type j. Thus it is that Miner's Rule allows the computation of characteristic life under a random usage spectrum $\Lambda(U)$ as

$$\beta = (EL) / \sum_{j=1}^{k} \frac{En_j(\Lambda)}{\beta_j} . \tag{2.3}$$

This conclusion has been pointed out previously in a less general context in [11].

The great utility of the modified Miner's Rule, as expressed in (2.3), coupled with Assumptions A° and B° is the statistical quantification of scatter factor. Suppose that characteristic life β as calculated by (2.3) is determined, then with α estimated from prior data by the methods presented in [10] we set the safe life t_ε at $100(1-\varepsilon)\%$ confidence where we take ε to be small. Thus t_ε is the life before which failure will occur with probability ε and is given by

$$t_\varepsilon = \beta[F^{-1}(\varepsilon)]^{1/\alpha}. \tag{2.4}$$

Hence the reciprocal of the scatter factor in (2.4) is $[F^{-1}(\varepsilon)]^{1/\alpha}$ which value can be determined for many choices of F and the most stringent one taken. Or alternatively, the usually applied scatter factor can be assessed as to the implied level of confidence by the use of (2.4).

4. Conclusion

In this expository note we have tried to point out the ubiquity of the Miner-Palmgren Rule for the calculation of fatigue life by showing that from a rather general and plausible, mathematical and probabilistic framework it appears as the expected value of stochastic fatigue life. These results explain its empirically verified utility and the difficulty of supplanting it with other "rules". We also show how such a result fits nicely in theory into any two parameter shape and location model for the calculation of safe service life.

Lastly, we do not pretend that any particular theory, such as this, can be the final word or that it is impossible that further knowledge of the physics of material could vitiate the model we have employed, such as α being a constant of the material. However, we do think that it does indicate that deterministic models for fatigue should be reassessed.

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